Recap: Caporaso-Harris recursion formula

finite sequences of non-negative integers

$\alpha = (\alpha_1, \alpha_2, \ldots)$ assigned pts

$\beta = (\beta_1, \beta_2, \ldots)$ unassigned pts

$e_k = (0, \ldots, 0, 1)$

kt position

$|\alpha| := \sum_i \alpha_i$ without multiplicity

$I\alpha := \sum_i i \cdot \alpha_i$ with multiplicity

$\alpha' \leq \alpha :\Leftrightarrow \alpha_i' \leq \alpha_i \quad \forall i$

$$\binom{\alpha}{\alpha'} := \prod_i \binom{\alpha_i}{\alpha_i'}$$

$$I^{\alpha} := \prod_i i^{\alpha_i}$$ product of multiplicities

$D \subset \mathbb{P}^2$ a fixed line and $I\alpha + I\beta = d$

$N^{d,\delta}(\alpha, \beta) :=$ # complex reduced plane curves of degree d and genus $\binom{d-1}{2} - \delta$ that

" relative Severi degree

- intersect D in α_i <u>fixed</u> pts with multiplicity i $\forall i \geq 1$
- intersect D in β_i <u>arbitrary</u> pts with mult i $\forall i \geq 1$
- pass through $\binom{d+1}{2} - \delta + |\beta|$

Caporaso - Harris's recursion formula

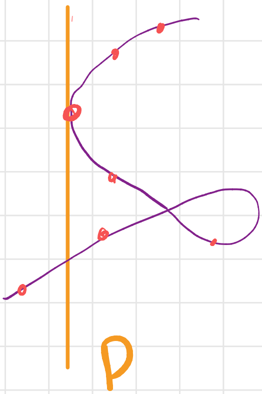
$$N^{d,\delta}(\alpha,\beta) = \sum_{k:\beta_k>0} k \cdot N^{d,\delta}(\alpha+e_k, \beta-e_k)$$

assign one more pt

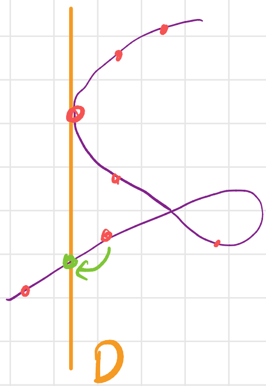
$$+ \sum_{\substack{\alpha'\leq\alpha \\ \beta'\geq\beta \\ \delta-\delta'+|\beta'-\beta|=d-1 \\ \delta'\leq\delta}} I^{\beta'-\beta} \binom{\alpha}{\alpha'} \binom{\beta'}{\beta} N^{d-1,\delta'}(\alpha',\beta')$$

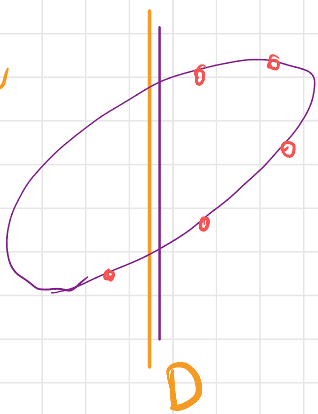
curve degenerates into C' of degree $d-1$ and D

Ex: $d=3, \delta=1, \alpha=(0,1), \beta=(1)$



move one point to D:





$$N^{3,1}((0,1),(1))$$
$$= N^{3,1}((1,1),0)$$
$$+ \binom{2}{1} N^{2,0}((0),(2))$$

Today we will see the Caporaso-Harris recursion formula for

- $S = \mathbb{P}^2$
- $S = \Sigma_m$ Hirzebruch surface
- $S = \mathbb{P}(1,1,m)$ weighted proj space

and prove it for tropical curves with quantum multiplicities [Gathmann-Markwig, Bloch-Göttsche]

and we will do some computations with floor diagrams.

$\rightsquigarrow$ node polynomials

Will replace • $S = \mathbb{P}^2$ by $S = \Sigma_m$

or $S = \mathbb{P}(1,1,m)$

• and d $\left(= \mathcal{O}_{\mathbb{P}^2}(d) = d \cdot H\right)$

class of a line in $\mathbb{P}^2$

by a line bundle on S / class in $H^2(S,\mathbb{Z})$

and count curves $\in |L| = \mathbb{P}H^0(S,L)$

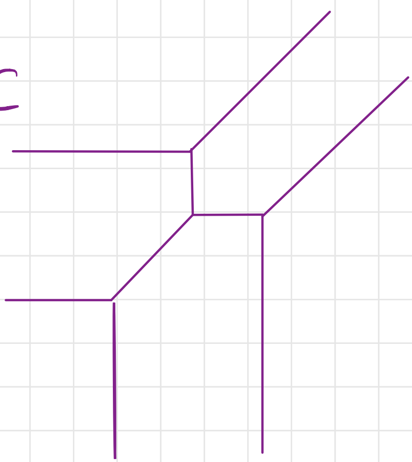
1^{st} case $S = \mathbb{P}^2$, $L = d \cdot H$

H = class of a line in $\mathbb{P}^2$

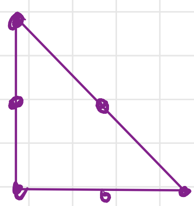
$$\Delta = \Delta_d = \operatorname{conv}\left((0,0), (0,d), (d,0)\right)$$

Ex:

C



degree (C) =



$= \Delta_2$

2^{nd} case: $S = \Sigma_m = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-m))$

H = class of a section with $H^2 = m$

F = class of a fiber

$L = d \cdot H + cF$

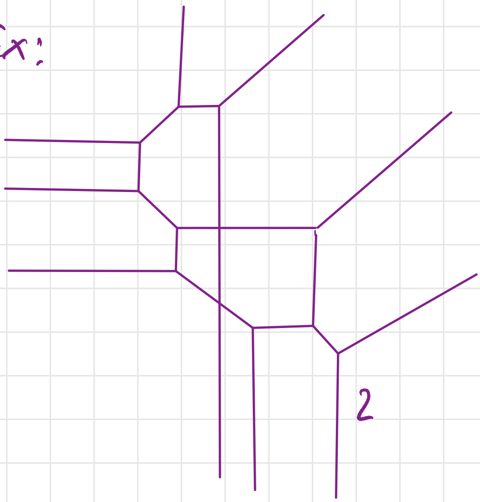
$E^2 = -m$

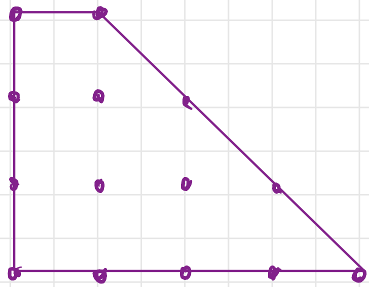
$\downarrow$

E, F

$H = E + mF$

Ex:





$d = 3, \quad c = 1, \quad m = 1$

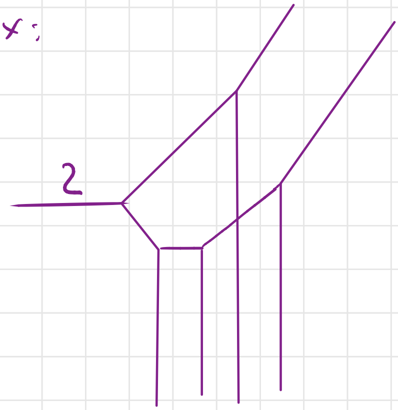
$$\Delta = \operatorname{conv}((0,0), (0,d), (c,d), (c+md, 0))$$

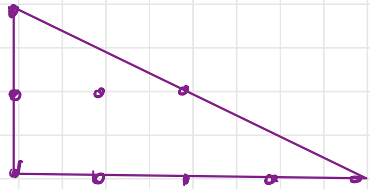
3^{rd} case: $S = \mathbb{P}(1,1,m)$

H = class of a line $H^2 = m$

$L = d \cdot H$

Ex:





$d = m = 2$

$\Delta = \mathrm{conv}\big((0,0), (0,d), (dm,0)\big)$

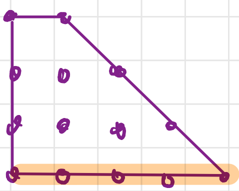
Δ = lattice polygon

$h : C \to \mathbb{R}^2$ parametrized tropical curve with degree Δ

D = an edge of Δ (D replaces the fixed line in the classical set up)

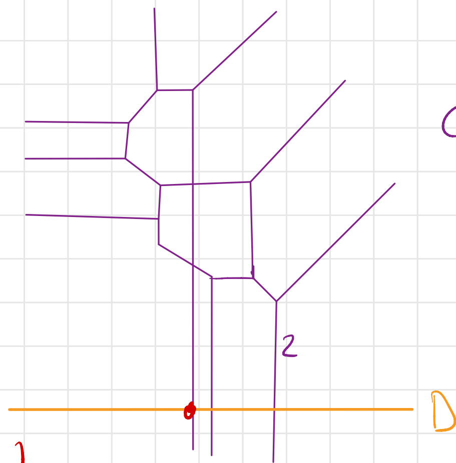
Def: • tropical boundary divisor of D ($=: \mathcal{D}$) is a classical line in $\mathbb{R}^2$ parallel to D and sufficiently far away dual to D (all intersections with C are orthogonal)

- C is tangent to D of order β if β "counts" the unbounded edges orthogonal to D



$= \Delta$

D



$\alpha = (1)$
$\beta = (1,1)$ Here $\quad \beta = (2,1)$

Let α, β st $I\alpha + I\beta = \text{length}(D)$

$\Pi :=$ $\quad n = \#(\Delta \cap \mathbb{Z}^2) - 1 - \delta$
$\quad\quad - I\alpha - I\beta + |\alpha| + |\beta|$

$\{p_1, \ldots, p_n\}$

tropically generic pts with precisely $|\alpha|$ pts on D

Def: C is (α, β)-tangent to D if $\alpha_i + \beta_i$ unbounded edges of C are orthogonal to D and have mult i and α_i of these pass through $D \cap \Pi$

Recall: mult $C = \prod_{\substack{\text{3-valent} \\ \text{vertices}}} 2\,\mathrm{Area}(\Delta_v)$

triangle dual to v in dual subdivision

refined multiplicity

$$\mathrm{mult}(C, y) = \prod_{\substack{\text{3-valent} \\ \text{vertices}}} [2\,\mathrm{Area}(\Delta_v)]_y$$

where $[i]_y = \dfrac{y^{i/2} - y^{-i/2}}{y^{1/2} - y^{-1/2}}$

refined relative multiplicity

$$\mathrm{mult}_{\alpha,\beta}(C,y) = \frac{1}{\prod_{i\geq 1}([i]_y)^{\alpha_i}} \cdot \mathrm{mult}(C,y)$$

refined relative Severi degree $N^{\Delta,\delta}(\alpha,\beta)(y)$

:= # δ-nodal tropical curves of degree Δ passing through Π that are (α,β)-tangent to D counted with $\mathrm{mult}_{\alpha,\beta}(C,y)$

Rmk: This depends on the choice of D.

Thm 7.3 in Bloch-Göttsche:

$N^{\Delta,\delta}(\alpha,\beta)(y)$ is independent of the choice of Π (as long as generic.)

Thm 7.5 Bloch-Göttsche / Thm 4.3 Gathmann-Markwig

Let Δ be as in one of the three cases above

For $S = X(\Delta)$ $\leftarrow \mathbb{P}^2, \Sigma_m, \mathbb{P}(1,1,m)$

$L = L(\Delta)$

$$\underset{\Delta}{\widetilde{N}}^{(S,L),\delta}(\alpha,\beta)(y) = \sum_{k:\,\beta_k>0} [k]_y \cdot N^{(S,L),\delta}(\alpha+e_k,\beta-e_k)(y)$$

$$+ \sum_{\substack{\beta'\geq\beta \\ \alpha'\leq\alpha}} \prod_i [i]_y^{\beta_i'-\beta_i} \binom{\alpha}{\alpha'}\binom{\beta'}{\beta} N^{(S,L-H),\delta'}(\alpha',\beta')(y)$$

$\delta' = \delta - H(L-H) + |\beta'-\beta|$

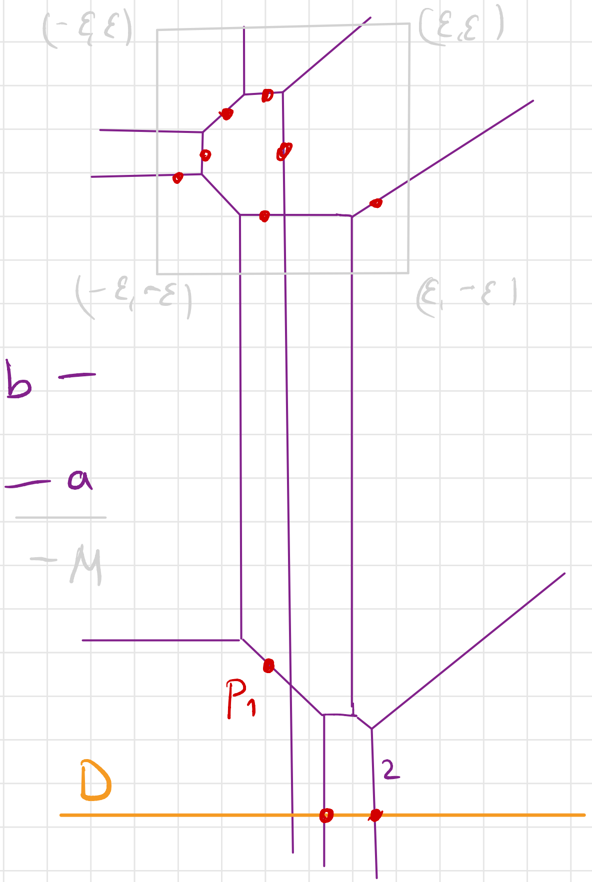
$I\alpha' + I\beta' = H(L-H)$

$\varepsilon > 0$, $M >> 0$, $\Pi = \{p_1, \dots, p_n\}$ st

1) x-coord of $p_i \in (-\varepsilon, \varepsilon)$

2) p_1 is not on D and its y-coord is $< -M$

3) all $p_i \neq p_1$ not lying on D have y-coord in $(-\varepsilon, \varepsilon)$



Lemma (Gathmann-Markwig)

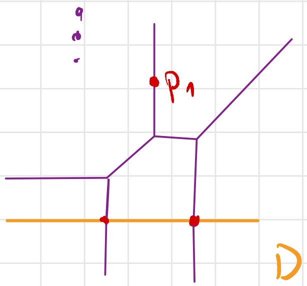
1) all vertices have x-coord $\in (-\varepsilon, \varepsilon)$

2) $\exists\ -M < a < b < -\varepsilon$ st in $\mathbb{R} \times [a, b]$ all edges of C are vertical

Pf of Recursion formula:

Case 1: p_1 lies on a vertical edge of weigth k

$\Rightarrow$ all edges with y-coord $\leq -\varepsilon$ are vertical:

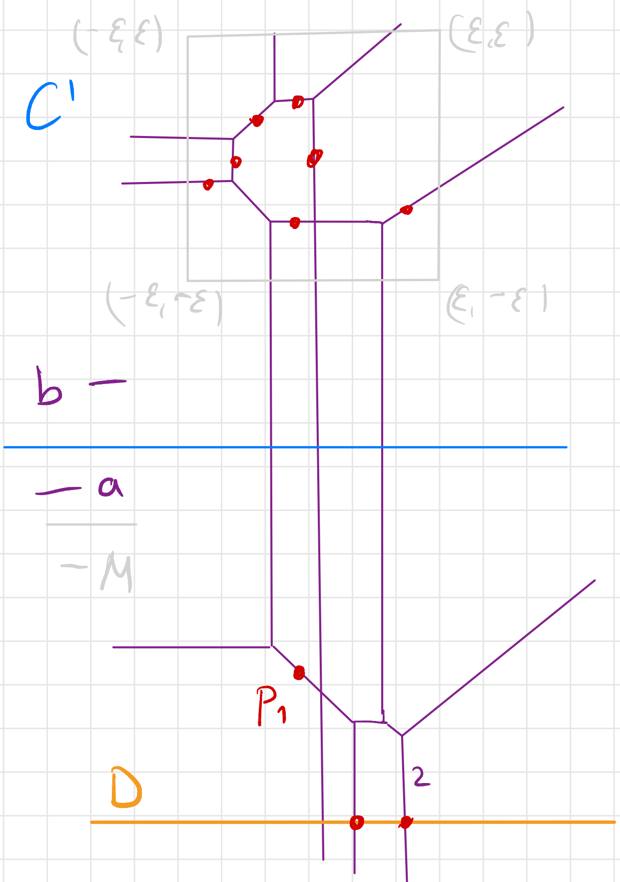


By Mikhalkin Lemma 4.20 one cannot go from one unbounded edge to another without passing a marking

$\Rightarrow$ can move p_1 to D

$$\mathrm{mult}_{\alpha,\beta}(C,y) = [k]_y\ \mathrm{mult}_{\alpha+e_k,\beta-e_k}(C,y)$$

$$\rightsquigarrow \sum_{k:\beta_k>0} [k]_y\ N^{(S,L),s}(\alpha+e_k,\beta-e_k)(y)$$



Case 2: p_1 does not lie on a vertical unbounded edge:

Can divide C into upper C' and lower part

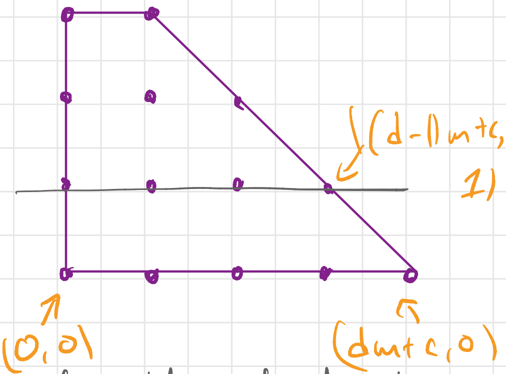
C' is a tropical curve that is

$\binom{\alpha}{\alpha'} \leftarrow \alpha' \leq \alpha$ $\qquad$ (α', β') - tangent to D.

$\binom{\beta'}{\beta} \leftarrow \beta' \geq \beta$

C' has degree Δ'

obtained from removing the bottom strip of Δ



To show $\quad I\alpha' + I\beta' =$ length of bottom edge of Δ'

$$\overset{?}{=} H(L-H)$$

and $\Delta' \leftrightarrow (S, L-H)$

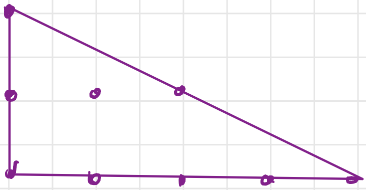
$S = \mathbb{P}^2$: $\quad \underset{\text{class of a line}}{H}\,(\underset{d \cdot H}{L}-H) = d-1$

$$\Delta = \Delta_d$$
$$\Delta' = \Delta_{d-1}$$

$S = \Sigma_m$: $\quad \underset{H^2 = m}{H}\,(\underset{dH + cF}{L} - H) = md + c - m$

$$= (d-1)m + c$$

$S = \mathbb{P}(1,1,m)$: $\quad \underset{H^2 = m}{H}\,(\underset{d \cdot H}{L} - H) = dm - m = (d-1)m$



$d = m = 2$

$\delta - \delta' = \#$ parallelograms in bottom strip of Δ in dual subdivision Δ_C

$= \#$ unbounded edges of C' that intersect D and are also unbounded in C

$= \underbrace{\#\text{ unbounded edges of } C' \text{ that intersect } D}_{\text{length of lower edge of } \Delta'}$

$\quad - \#$ edges in C' that become bounded in C

$= H(L-H) - |\beta' - \beta|$

Multiplicities:

$$\mathrm{mult}_{\alpha,\beta}(C,y) = \frac{1}{\prod_i ([i]_y)^{\alpha_i}} \, \mathrm{mult}(C,y)$$

$$= \frac{\prod_i ([i]_y)^{\cancel{\alpha_i} - \cancel{\alpha_i'} + \beta_i' - \beta_i}}{\prod_i ([i]_y)^{\cancel{\alpha_i}\, \alpha_i'}} \quad \mathrm{mult}(C', y)$$

$$= \prod_i ([i]_y)^{\beta_i' - \beta_i} \, \mathrm{mult}_{\alpha', \beta'}(C, y)$$

$\square$

Rmk: Gathmann–Markwig give another proof ($S = \mathbb{P}^2$, $S = \mathbb{P}^1 \times \mathbb{P}^1$) using a relative version of λ-increasing paths.

<u>Floor diagrams</u> (Block – relative node polynomials for plane curves)

Now $S = \mathbb{P}^2$

<u>Prop 7.7</u> (Block–Göttsche)

$$N^{d,\delta}(\alpha, \beta)(y) = \sum_{D \in FD(d,\delta)} \mathrm{mult}_\beta(P, y)\, \nu_{\alpha,\beta}(D)$$

Need to define

- $FD(d, \delta)$
- $\mathrm{mult}_\beta(D, y)$
- $\nu_{d,\beta}(D)$

Def floor diagrams = directed graphs on $\{1, \dots, d\}$ and weights $w(e) \in \mathbb{Z}_{>0}$

st 1) $i \to j \Rightarrow i < j$

2) $\mathrm{div}(j) := \sum_{j \xrightarrow{e} k} w(e) - \sum_{i \xrightarrow{e} j} w(e) \leq 1$

degree $d(D) := d$

genus $g(D) :=$ genus of the graph

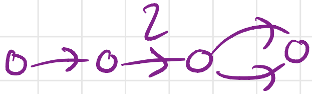
cogenus $\delta(D) := \binom{d-1}{2} - g$ if D is connected

Otherwise

$$\delta(D) = \sum \delta_j + \sum_{j<j'} d_j d_{j'}$$

$FD(d, \delta) = \{$ floor diagrams of degree d and cogenus $\delta \}$

Ex:



$d = 4$

$g = 1$

$\delta = 2$

div	1	1	0	-2

$$\mathrm{mult}(D, y) := \prod_{\text{edges}} \left([w(e)]_y\right)^2$$

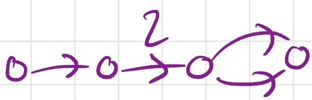
$$\mathrm{mult}_\beta(D, y) := \prod_{i \geq 1} \left([i]_y\right)^{\beta_i} \cdot \mathrm{mult}(D, y)$$

Def: (α, β)-marking of a floor diagram D, $I\alpha + I\beta = d$

Step 1: Fix $\{\alpha^i\}, \{\beta^i\}$ $i = 1, \dots, d$

st 1) $\sum \alpha^i = \alpha$ $\sum \beta^i = \beta$

2) $I\alpha^i + I\beta^i = 1 - \mathrm{div}(i)$ $\forall i$



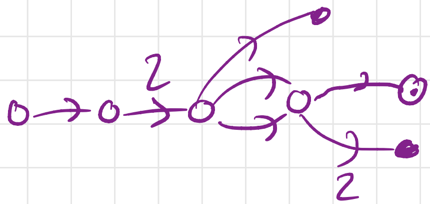
$\alpha = (1)$

$\beta = (1, 1)$

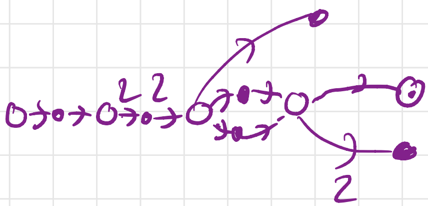
div	1	1	0	-2
1-div	0	0	1	3
α^i	0	0	0	(1)
β^i	0	0	(1)	(0,1)

Step 2: i = vertex

$\forall j$ create β^i_j and α^i_j new vertices and connect them to i with an edge of weight j directed away from i



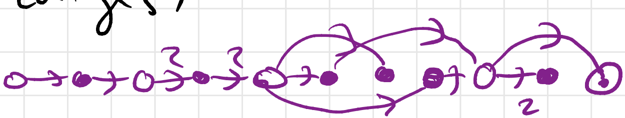
Step 3: Subdivide each edge in original graph



$=: \tilde{D}$

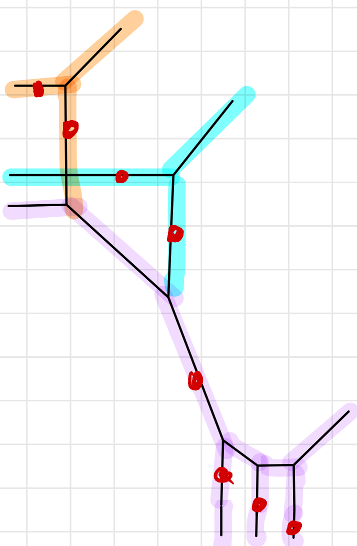
Step 4: Linearly order vertices of $\tilde{D}$ st $i \to j \Rightarrow i < j$ and st α-vertices are largest

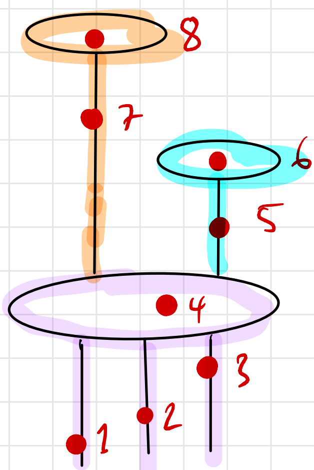
$v_{\alpha,\beta}(D) = 5$



$\nu_{(\alpha,\beta)}(D) := \#\{(\alpha,\beta)\text{-marking}\} \Big/ \left(\begin{array}{l}\text{weight}\\ \text{preserving}\\ \text{auto}\\ \text{that fixes}\\ D\end{array}\right)$

Rmk: This looks different from the floor diagrams we have seen before.





New floor diagram

Thm 7.8 (Block-Göttsche)

For $\delta \geq 1$ there is a polynomial $N_\delta(\alpha, \beta, y) \in \mathbb{Q}[y^{\pm 1}][\alpha_1, \dots, \alpha_\delta, \beta_1, \dots, \beta_\delta]$ st $\forall \alpha, \beta$ with $|\beta| > \delta$ we have

$$N^{d,\delta}(\alpha, \beta)(y) = \prod_i ([i]_y)^{\beta_i} \frac{(|\beta|-\delta)!}{\beta_1! \, \beta_2! \cdots} N_\delta(\alpha, \beta, y)$$

Ex (non-refined):

- $N_0(\alpha, \beta)(y) = 1$
- $N_1(\alpha, \beta) = 3d^2 |\beta| - 8d|\beta| + d\beta_1 + |\beta|\alpha_1 + |\beta|\beta_1 + 4|\beta| - \beta_1$